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Statistical Modeling of Unidirectional Fibrous Structures

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Summary: Some textiles such as slivers, rovings, yarns, and highly oriented polymer fibers as well as the reinforcing structure of unidirectional composites have a kind of unidirectional or quasi-unidirectional fibrous structures. The statistical properties of their structure and strength can be modeled by using idealized fiber bundles as model elements. In this study the tensile test process of unidirectional short fiber structures is modeled for different damage types using the instantaneous fracture model and special idealized fiber bundles for gradual damages such as fiber breakage and fiber slippage. Constant fiber length and exponential fiber length distribution as extreme cases of the Erlang distributions were used for analysis. In case of exponential fiber length distribution and constant fiber breaking strain simple analytical relationships between the mean tensile strength and the fiber length were derived and compared to those for constant fiber length and written in a general form which is valid for all the damage modes discussed. The convex linear combination of the solutions for exponential fiber length distribution and constant fiber length was proposed to use for cases when the variation coefficient of the fiber length is between 0 and 1. The practical applicability of the results was demonstrated by identifying the relationship between the mean tensile strength and the average molecule mass of polypropylene fibers that made it possible to estimate the critical molecule mass and the tensile strength of the molecules without further measurements.

Keywords: fiber bundle; modeling; statistical mechanics; strength; unidirectional

Introduction

A number of textiles such as slivers, rovings, yarns and highly oriented polymer fibers (HPPE, Kevlar, LCP polyester, etc.) as well as the reinforcing structure of unidirectional composites have a kind of unidirectional or quasi-unidirectional fibrous structure. It is well known that certain small assemblies called fiber bundles play an important role in the macroscale mechanical properties of the fibrous structures as intermediate structural elements[1–8]. The statistical properties of their structure and strength can be modeled by using idealized statistical fiber bundles, kinds of fiber-bundle-cells as model elements (Figure 1)[3,4,8].

The fibers of these bundles are supposed to be perfectly flexible, linearly elastic and to break at a random strain. They are straight in the E-bundle, loose or pretensioned in the EH-bundle, and slack in the ET-bundle, and gripped ideally in these cases. The fibers in the ES-bundle are straight but they may slip out of their grip or create fiber-chains with slipping bonds. All the parameters can be probabilistic variable and they are considered statistically independent.

In comparison with other methods such as the finite element method (FEM) and various simulation techniques[9–12] the statistical fiber-bundle-cells method works without meshing and can make it possible to derive simple analytical formulas.

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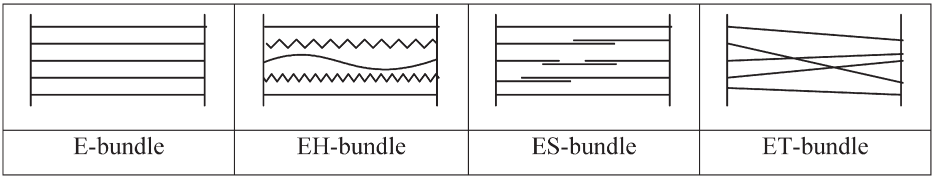

Figure 1.
Idealized fiber-bundle-cells.

In this study the tensile test process of unidirectional or highly oriented short fiber structures are modeled and simple relationships between the tensile strength and the fiber length assumed to be of exponential distribution are presented for different damage types taking into account both the fiber breakage and the fiber slippage. The relationships are compared with those obtained for constant fiber length earlier[8] and an estimation method is proposed for some general cases. The applicability of the results is demonstrated by estimating and analyzing the strength of PP fibers as a function of the average molecule mass.

Concept of Modeling with Fiber Bundles

Breakages and Slippages of Fibers

A unidirectional fibrous structure forms a so called fiber flow[1] in which the critical adhesion length of fibers is the half of the critical fiber length ($l_S = l_{crit}/2$) and determined by the specific adhesion force (f_b) and the mean fiber breaking load $F_S = f_b l_S$. It is assumed that the length of the structure as a gauge length is much greater than the mean fiber length and there is a kind of adhesion between the fibers as well as subjecting the system to increasing tensile load the surface of fracture is a plane cross section. During a tensile test the fibers intersecting the breaking cross section break or slip out of the grip created by the other fibers in their vicinity (Figure 2) depending on whether their minimal part-length (l_m) on the two sides of the cross section is larger than l_S or not.

Fiber Length Distributions of Different Types

The overall fiber length (l) distribution function (FLD) concerning any fiber in the fiber system is given by the following probability ($x \geq 0$):

$$Q_l(x) = P(l < x) \tag{1}$$

The lengths of the left (l^-) and right (l^+) hand parts of a fiber intersecting the breaking cross section are called beard lengths (Figure 3).

The beard length distribution of fibers is determined by the FLD[1]:

$$S(x) = P(l^- < x) = P(l^+ < x) = \int_o^x \frac{1 - Q_l(u)}{\bar{l}} du \tag{2}$$

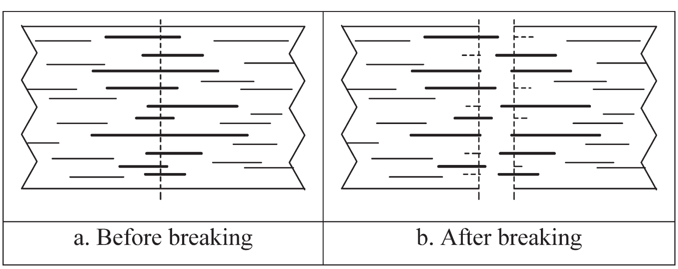

Figure 2.
Arrangement of the fibers intersecting the breaking cross section.

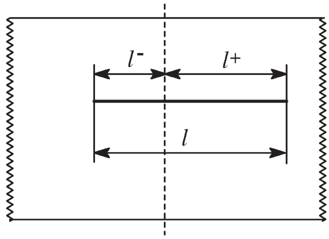

Figure 3.
Protruding parts of a fiber intersecting the breaking cross section.

The smaller one of the beard lengths (l_m) called active beard length determines whether the given fiber intersecting the breaking cross section will slip out ($l_m < l_S$) or break ($l_m \geq l_S$):

$$l_m = min(l^-, l^+) \tag{3}$$

The distribution function of the active beard length has been derived[1] using geometrical considerations based on a simple transformation of the beard distribution:

$$S_m(x) = P(l_m < x) = S(2x) \tag{4}$$

Application of Erlang's Distributions

The Erlang distribution is a special gamma distribution the density function of which is as follows (Figure 4)[13]:

$$q_l(x) = \frac{n^n}{m(n-1)!}\left(\frac{x}{m}\right)^{n-1} e^{-\frac{x}{m}} \tag{5}$$

where n is a positive integer, $m = \bar{l}$ is the expected value. The standard deviation is $m/\sqrt{n}$, that is it depends on n and tends to 0 when $n \to \infty$.

In case of $n = 1$ Equation (5) gives the exponential distribution and if $n \to \infty$ under the condition that the expected value remains constant the limit is a degenerated distribution with zero deviation which means that the fiber length is a constant value. In this sense the exponential fiber length distribution and the constant fiber length realize kinds of extreme cases (Figure 4).

The exponential fiber length (l) distribution (EFLD) ($x \geq 0$) is given by:

$$Q_l(x) = P(l < x) = 1 - e^{-\frac{x}{\bar{l}}} \tag{6}$$

In case of constant fiber length (l_o) the fiber length distribution is degenerated and can be described with a unit step function ($1(x)$) ($x \geq 0$):

$$Q_l(x) = P(l < x) = 1(x - l_o) = \begin{cases} 1, & 1, x \geq l_o \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

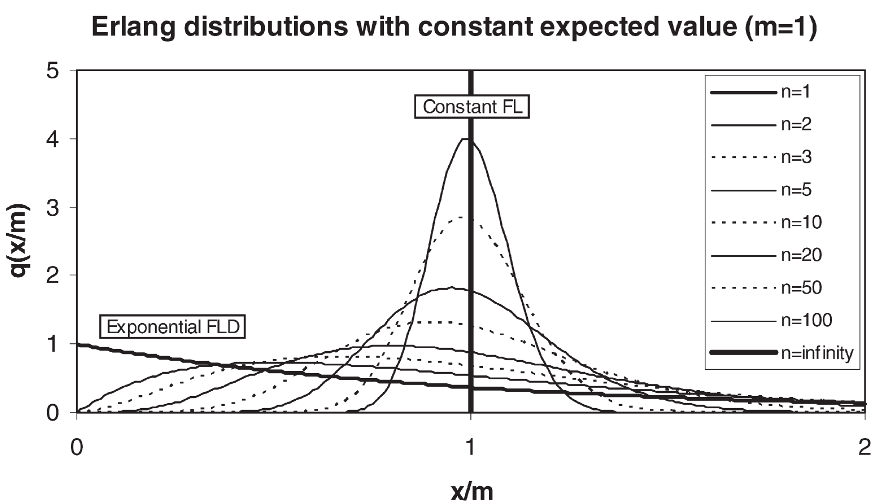

Figure 4.
Erlang distribution family.

In case of EFLD Equation (2) leads to a result identical to the EFLD:

$$S(x) = 1 - e^{-\frac{x}{l}} \quad (8)$$

The beard length distribution for constant fiber length is a uniform distribution in $(0, l_o)$:

$$S(x) = \min(x/l_o, 1) = \begin{cases} 0, & x \leq 0 \\ x/l_o, & 0 < x < l_o \\ 1, & x \geq l_o \end{cases} \quad (9)$$

In case of exponential FLD the active beard length distribution is identical to the EFLD again:

$$S_m(x) = 1 - e^{-\frac{2x}{l}} \quad (10)$$

The active beard length distribution for constant fiber length is a uniform distribution as well, however with half a range:

$$S_m(x) = \min(2x/l_o, 1) = \begin{cases} 0, & x \leq 0 \\ 2x/l_o, & 0 < x < l_o/2 \\ 1, & x \geq l_o/2 \end{cases} \quad (11)$$

Special ES-Bundles

Under tensile load the slippages and breakages of fibers takes place gradually one after another determining a kind of damaging or breaking process characterized by the changes in the tensile force measured. The limit load (F_b) of slippage is proportional to l_m ($F_b = f_b l_m$) and so is the deformation (ε_b) related to F_b. This behavior can be described with special versions of the ES-bundle because the slippage force and length are independent in the original ES-bundle while they are in relation in this case. During the slippage of a fiber the resistance may be constant (Coulomb friction – ES1-bundle) or decrease (like secondary bonds between chain-molecules – ES2-bundle). Accordingly, two kinds of ES-bundle can be defined (Figure 5) where every fiber either breaks or slips out and $0 \leq \alpha \leq 1$ is the slippage length factor giving the ratio of l_m along which the fiber produces resistance against its pulling out. The diagrams in Figure 5 represent the characteristic of the ES1 and ES2-bundles[8]. The only difference between them is that during the slipping out the slippage resistance force remains constant in case of the ES1-bundle while it decreases gradually in the case of the ES1-bundle.

Modeling the fibers intersecting the breaking cross section of the fiber system with an ES1-bundle the one fiber related tensile force process of the fiber system can be described by[8]:

$$\overline{F}_{ES1}(u) = Ku(1 - Q_{\varepsilon_S}(u))(1 - Q_{\varepsilon_b}(u)) + K \int_{\frac{u}{1+\alpha}}^{u} x(1 - Q_{\varepsilon_S}(x))dQ_{\varepsilon_b}(x) \quad (12)$$

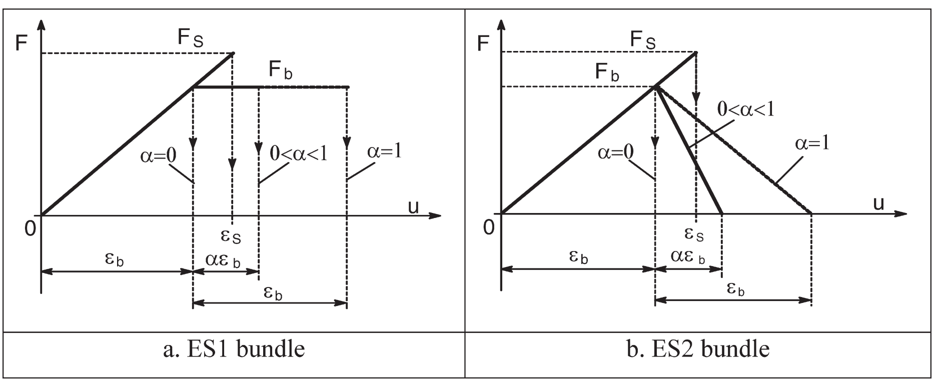

Figure 5.
Load-strain (F-u) characteristic of fibers in the special versions of the ES bundle.

where ε_b is the relative slippage length of fibers:

$$Q_{\varepsilon_b}(x) = S_m\left(\frac{K}{f_b}x\right) \tag{13}$$

In the case of the ES2-bundle we obtain[8]:

$$\begin{aligned}\overline{F}_{ES2}(u) &= \overline{F}_{ES1}(u) - \frac{K}{\alpha}\int_{\frac{u}{1+\alpha}}^{u}[u-x](1-Q_{\varepsilon_S}(x))dQ_{\varepsilon_b}(x)\\ &= Ku(1-Q_{\varepsilon_S}(u))(1-Q_{\varepsilon_b}(u)) + \frac{K}{\alpha}\int_{\frac{u}{1+\alpha}}^{u}[(1+\alpha)x-u]\\ &\quad\times(1-Q_{\varepsilon_S}(x))dQ_{\varepsilon_b}(x)\end{aligned} \tag{14}$$

The first term in the formulas as per Equations (12) and (14) describes the expected value process of the tensile force up to the breakages or the beginning of the slippages while the second term gives the effect of slippages.

It is easy to see that in the cases of $\alpha \to 0$ and $\alpha \to \infty$ the limit expressions for both the ES1 and the ES2 bundle are respectively as follows:

$$\lim_{\alpha\to 0}\overline{F}_{ES1}(u) = \lim_{\alpha\to 0}\overline{F}_{ES2}(u) = Ku(1-Q_{\varepsilon_S}(u))(1-Q_{\varepsilon_b}(u)) \tag{15}$$

$$\lim_{\alpha\to\infty}\overline{F}_{ES1}(u) = \lim_{\alpha\to\infty}\overline{F}_{ES2}(u) = Ku(1-Q_{\varepsilon_S}(u))(1-Q_{\varepsilon_b}(u)) + K\int_{o}^{u}x(1-Q_{\varepsilon_S}(x))dQ_{\varepsilon_b}(x) \tag{16}$$

The tensile strength of the fibrous structure is defined by the maximum of the expected value process of the tensile force during the tensile test.

Modeling the Tensile Strength of Unidirectional Fibrous Structures

The modeling of the tensile strength was performed for two basic damage modes that is the simultaneous and gradual damage modes. In order to develop simple formulas the calculations were performed for constant fiber length (l_o) and exponential fiber length distribution as well as the fiber breaking strain (ε_S) is considered constant therefore its distribution function has a form similar to Equation (7).

Instantaneous and Simultaneous Damages

In this case it is assumed that all the slippages (S) and breakages (B) take place instantaneously and at the same time creating an extreme damage mode.

General Formulation

In the case of instantaneous damages the mean tensile strength of the fiber system related to one fiber is given by[8]:

$$F* = f_b\int_{o}^{l_S}xdS_m(x) + F_S(1-S_m(l_S)) \tag{17}$$

where f_b is the specific adhesion resistance and F_S, ε_S, and K are the mean tensile strength, the mean breaking strain, and the tensile stiffness of fibers, respectively. After normalizing Equation (17) with F_S we obtain:

$$\frac{F*}{F_S} = \frac{1}{l_S}\int_{o}^{l_S}xdS_m(x) + 1 - S_m(l_S) \tag{18}$$

where Equation (19) was put to use as well:

$$F_S = K\varepsilon_S = f_b l_S \tag{19}$$

Special Cases of Fiber Length Distributions

Using Equation (4) for calculating the normalized strength of fiber system with exponential FLD yields:

$$\frac{F*}{F_S} = \frac{\bar{l}}{2l_S}\left[1-e^{-\frac{2l_S}{\bar{l}}}\right] = \frac{\lambda}{2}\left[1-e^{-\frac{2}{\lambda}}\right] \tag{20}$$

where $E(l) = \bar{l}$ is the mean fiber length, l_S is the critical adhesion length, and λ is the normalized fiber length:

$$\lambda = \frac{\bar{l}}{l_S} \tag{21}$$

In case of constant fiber length Equation (18) has the following form[8].

$$\frac{F*}{F_S} = \begin{cases} \frac{1}{4}\frac{l_o}{l_s}, & l_o/l_S < 2 \\ 1 - \frac{l_s}{l_o}, & 2 \le l_o/l_S \end{cases} \quad (22)$$

Figure 6 shows the plots of the mean tensile strength calculated with Equations (20) and (22) as a function of the mean fiber length.

The asymptotic values are identical for the two curves, while for finite fiber length ($0 < \bar{l}/l_S < \infty$) the exponential FLD gives larger strength values than that for the constant FL. The maximum difference between them is about 19 percent at $\bar{l}/l_S = 1.2$. All that means that in case of instantaneous and simultaneous damages the positive standard deviation in fiber length is advantageous concerning the tensile strength of the unidirectional fiber system.

Gradual Damage Process

In the damage process of a real fibrous structure the slippages and breakages take place successively and gradually due to the increasing tensile load. To model this process, special versions of the ES-bundle denoted with ES1 and ES2 are used.

Constant Fiber Length

According to the analysis carried out in case of constant fiber length earlier the relationship between the normalized bundle breaking force and normalized fiber length can be given by a simple general formula for all the damage modes discussed above (instantaneous and gradual damages) as follows[8]:

$$\frac{F*(\lambda)}{F_S} = \begin{cases} \frac{\lambda}{4}C(\alpha), & \lambda < \frac{2}{C(\alpha)} \\ 1 - \frac{1}{C(\alpha)\lambda}, & \lambda \ge \frac{2}{C(\alpha)} \end{cases} \quad (23)$$

where C(α) is a constant depending on the slippage length factor:

$$C(\alpha) = \begin{cases} 1, & \text{instantaneous damages} \\ \frac{(1+\alpha)^2}{1+(1+\alpha)^2}, & \text{ES1} - \text{bundle} \\ \frac{1+\alpha}{2+\alpha}, & \text{ES2} - \text{bundle} \end{cases} \quad (24)$$

The constant (C) for instantaneous total breaking can be obtained by a limit-transition ($C(\alpha) \to 1$, $\alpha \to \infty$) in case of both the ES1 and the ES2 bundles. It is easy to use Equation (23) for identifying the relationship between the measured data of tensile strength and fiber length.

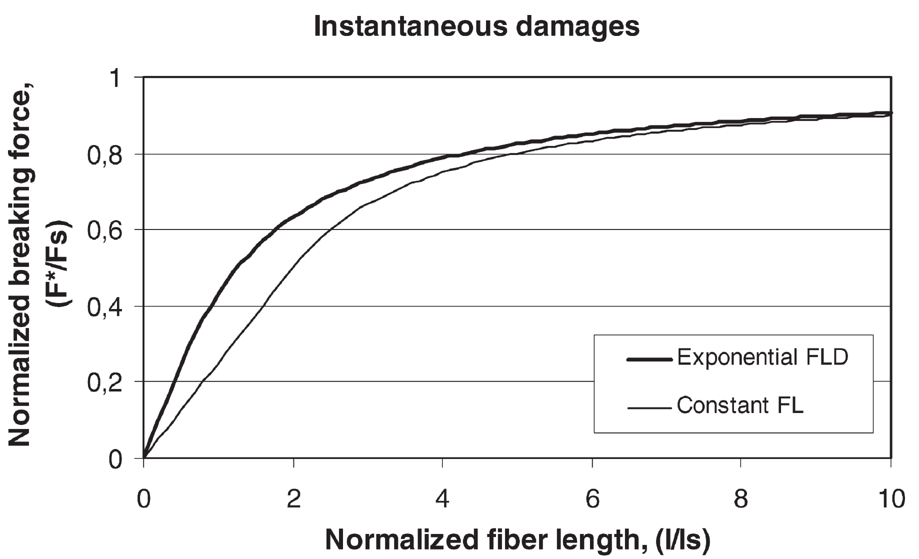

Figure 6.
Normalized mean tensile strength of the fiber system versus normalized fiber length in the case of instantaneous damages.

Exponential Fiber Length Distribution

For modeling the gradual damages ES1 and ES2 bundles and Equation (10) were used.

Application of the ES1-Bundle

Considering formula (14) for the ES1-bundle we obtain:

$$\bar{F}_{ES1}(u) = Ku[1 - 1(u-\varepsilon_S)]e^{-\frac{2Ku}{f_b l}} + KI_2 = \begin{cases} \frac{f_b\bar{l}}{2}\left[\left(1+\frac{2Ku}{f_b\bar{l}(1+\alpha)}\right)e^{-\frac{2Ku}{f_b l(1+\alpha)}} - e^{-\frac{2Ku}{f_b l}}\right], & \frac{u}{1+\alpha} < u \le \varepsilon_S \\ \frac{f_b\bar{l}}{2}\left[\left(1+\frac{2Ku}{f_b\bar{l}(1+\alpha)}\right)e^{-\frac{2Ku}{f_b l(1+\alpha)}} - \left(1+\frac{2K\varepsilon_S}{f_b l}\right)e^{-\frac{2K\varepsilon_S}{f_b l}}\right], & \frac{u}{1+\alpha} \le \varepsilon_S < u \\ 0, \quad \varepsilon_S < \frac{u}{1+\alpha} < u \end{cases} \tag{28}$$

To simplify the calculations, let us determine the integrals in Equations (14) and (16) at first:

$$I_1 = \int_{\frac{u}{1+\alpha}}^{u} u(1-Q_{\varepsilon_S}(x))dQ_{\varepsilon_b}(x) = u\int_{u_{\alpha,S}}^{u_S} dQ_{\varepsilon_b}(x) = u\left(e^{-\frac{2K}{f_b l}u_{\alpha,S}} - e^{-\frac{2K}{f_b l}u_S}\right) \tag{25}$$

where

$$u_S = \min(u, \varepsilon_S) \qquad u_{\alpha,S} = \min\left(\frac{u}{1+\alpha}, \varepsilon_S\right) \tag{26}$$

$$I_2 = \int_{\frac{u}{1+\alpha}}^{u} x(1-Q_{\varepsilon_S}(x))dQ_{\varepsilon_b}(x) = \int_{u_{\alpha,S}}^{u_S} x dQ_{\varepsilon_b}(x) = \frac{f_b\bar{l}}{2K}\left(\left(1+\frac{2Ku_{\alpha,S}}{f_b\bar{l}}\right)e^{-\frac{2K}{f_b l}u_{\alpha,S}} - \left(1+\frac{2Ku_S}{f_b\bar{l}}\right)e^{-\frac{2K}{f_b l}u_S}\right) \tag{27}$$

The following relationships together with Equation (19) give a hand to simplify Equation (28).

$$\lambda = \frac{\bar{l}}{l_S} = \frac{f_b\bar{l}}{F_S} \tag{29}$$

$$z = \frac{u}{\varepsilon_S} \tag{30}$$

Using Equations (19), (29) and (30) the variables in Equation (28) can be normalized with the mean breaking strength (F_S) and strain (ε_S) of fibers:

$$FH_1(z) = \frac{\bar{F}_{ES1}(z)}{F_S} = \begin{cases} \frac{\lambda}{2}\left[\left(1+\frac{2z}{\lambda(1+\alpha)}\right)e^{-\frac{2z}{\lambda(1+\alpha)}} - e^{-\frac{2z}{\lambda}}\right], & 0 \le z \le 1 \\ \frac{\lambda}{2}\left[\left(1+\frac{2z}{\lambda(1+\alpha)}\right)e^{-\frac{2z}{\lambda(1+\alpha)}} - \left(1+\frac{2}{\lambda}\right)e^{-\frac{2}{\lambda}}\right], & 1 < z \le 1+\alpha \\ 0, \quad 1+\alpha < z \end{cases} \tag{31}$$

Figure 7 shows the calculated average tensile load process as a function of the bundle strain at different slippage factors (α) and fiber lengths.

If the fiber length (l) is greater than the critical fiber length ($l_{crit} = 2l_S$) there is a drop in the tensile force when reaching the fiber breaking strain ($z = 1$) because of the breaking fibers. The value of a drop increases with the mean fiber length and so does the force with the slippage factor.

The strength of the fiber system is defined by the maximum value of the expected value of the tensile load which is determined using extreme value analysis. In the first section of the expected value process according to Equation (31) the maximum value gives the strength compo-

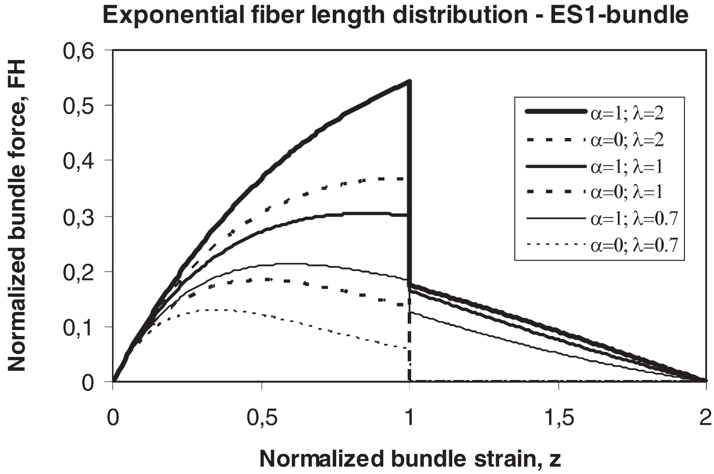

Figure 7.
Normalized tensile load versus normalized strain at different slippage factors and mean fiber length in case of ES1-bundle.

nent. The necessary condition of the existence of the maximum is that the derivative of Equation (31) equals zero ($0 \le z \le 1$).

$$0 = FH'(z) = \frac{2}{\lambda}\left[-\frac{2z}{\lambda(1+\alpha)^2}e^{-\frac{2z}{\lambda(1+\alpha)}} + e^{-\frac{2z}{\lambda}}\right] \quad (32)$$

By introducing the following notation:

$$x = \frac{2z}{\lambda} \quad (33)$$

Equation (32) can be written into a simpler form:

$$\frac{x}{(1+\alpha)^2} = e^{-x\frac{\alpha}{1+\alpha}} \quad (34)$$

Note, that Equation (34) is independent of λ, therefore the solution (x^*) of that depends on α only: $x^* = x^*(\alpha)$. With that the solution of Equation (33) has the following shape:

$$z^* = \frac{\lambda}{2}x^*(\alpha) \quad (35)$$

An invertible approximate solution of Equation (34) valid for $0 \le \alpha \le 1$ is as follows (see Appendix):

$$x^* = x^*(\alpha) = \frac{1+\alpha}{\alpha}y^* \approx \frac{1+\alpha}{\alpha} \times \frac{a_1}{2a_2}\left(\sqrt{1+4a_2\frac{\alpha(1+\alpha)}{a_1^2}} - 1\right) \quad (36)$$

where the coefficients are $a_1 = 0.7419$, $a_2 = 1.8567$. Equation (36) provides an approximation with a relative error smaller than 2%. z^* gives the place of the possible maximum in the first section of the mean force process. If $z^* > 1$ there is a drop at the beginning of the second section hence the maximum can be found at the end of the first section ($z^* = 1$).

$$z^* = \begin{cases} \frac{\lambda}{2}x^*(\alpha), & 0 \le z^* < 1 \\ 1, & 1 \le z^* \end{cases} = \min\left(\frac{\lambda}{2}x^*(\alpha), \quad 1\right) \quad (37)$$

Substituting Equation (37) into Equation (31) gives the relationship between the tensile strength of fiber system and the mean fiber length for the ES1-bundle (Figure 8).

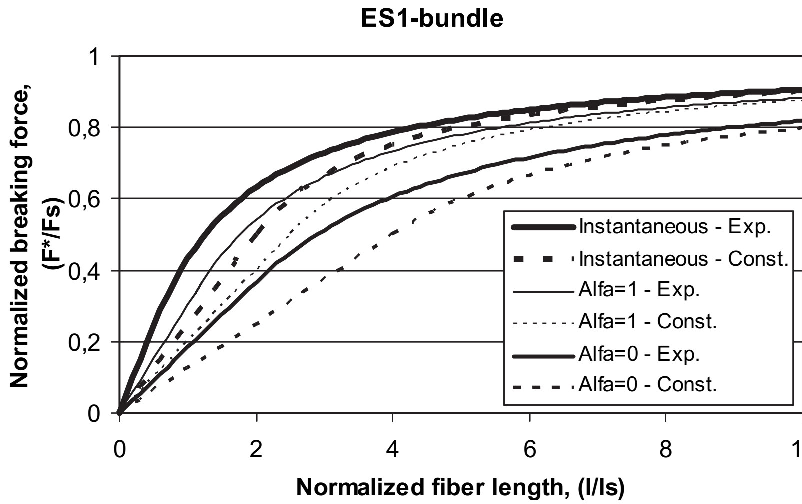

Figure 8.
Normalized breaking force modeled with ES1-bundle versus mean fiber length at different slippage factor in case of exponential FDL and constant FL.

$$FH_1* = FH_1(z*) = \begin{cases} \frac{\lambda}{2}\left[\left(1+\frac{x*(\alpha)}{1+\alpha}\right)e^{-\frac{x*(\alpha)}{1+\alpha}} - e^{-x*(\alpha)}\right], & \lambda < \frac{2}{x^*(\alpha)} \\ \frac{\lambda}{2}\left[\left(1+\frac{2}{\lambda(1+\alpha)}\right)e^{-\frac{2}{\lambda(1+\alpha)}} - e^{-\frac{2}{\lambda}}\right], & \lambda \geq \frac{2}{x^*(\alpha)} \end{cases} \quad (38)$$

In Figure 8 the normalized tensile strength of fibrous structures modeled with ES1 bundle is depicted as a function of the normalized fiber length at different slippage length factors (α) in the cases of exponential FLD and constant FL including the case of the instantaneous total breaking. It is visible that the curves for the gradual damages provide smaller strength values than those for the instantaneous damages and the strength values decrease with decreasing the slippage length represented by the slippage factor. On the other hand, in case of identical damage modes, the strength values for exponential FLD are greater than those for constant FL if $0 < \bar{l}/l_S < \infty$.

Application of the ES2-Bundle

On the basis of Equations (14), (25), (27), and (28) the mean tensile force for the ES2-bundle can be written easily:

$$\bar{F}_{ES2}(u) = \bar{F}_{ES1}(u) - \frac{K}{\alpha}(I_1 - I_2)$$
$$= \begin{cases} \frac{(1+\alpha)f_b\bar{l}}{2\alpha}\left(e^{-\frac{2Ku}{f_b\bar{l}(1+\alpha)}} - e^{-\frac{2Ku}{f_b\bar{l}}}\right), & \frac{u}{1+\alpha} < u \leq \varepsilon_S \\ \frac{(1+\alpha)f_b\bar{l}}{2\alpha}\left[e^{-\frac{2Ku}{f_b\bar{l}(1+\alpha)}} - e^{-\frac{2K\varepsilon_S}{f_b\bar{l}}} - \frac{2K\varepsilon_S}{f_b\bar{l}}\left(1-\frac{u}{\varepsilon_S(1+\alpha)}\right)e^{-\frac{2K\varepsilon_S}{f_b\bar{l}}}\right], & \frac{u}{1+\alpha} \leq \varepsilon_S < u \\ 0, \quad \varepsilon_S < \frac{u}{1+\alpha} < u \end{cases} \quad (39)$$

Similarly to that for ES1, Equation (39) can be normalized:

$$FH_2(z) = \frac{\bar{F}_{ES2}(z)}{F_S} = \begin{cases} \frac{1+\alpha}{\alpha}\frac{\lambda}{2}\left(e^{-\frac{2z}{\lambda(1+\alpha)}} - e^{-\frac{2z}{\lambda}}\right), & 0 \le z \le 1 \\ \frac{1+\alpha}{\alpha}\frac{\lambda}{2}\left[e^{-\frac{2z}{\lambda(1+\alpha)}} - \left(1 + \frac{2}{\lambda}\left(1 - \frac{z}{1+\alpha}\right)\right)e^{-\frac{2}{\lambda}}\right], & 1 < z \le 1+\alpha \\ 0, & 1+\alpha < z \end{cases} \quad (40)$$

The curves in Figure 9 are the calculated mean tensile force processes for the ES2-bundle. After the peak they contain a part concave from below caused by the decreasing slippage resistance.

In the case of ES2-bundle strength one can proceed like that for the ES1-bundle. The necessary condition for the maximum in the first section of Equation (40) is as follows:

$$0 = FH'(z) = \frac{2}{\lambda}\left[-\frac{2}{\lambda(1+\alpha)}e^{-\frac{2z}{\lambda(1+\alpha)}} + \frac{2}{\lambda}e^{-\frac{2z}{\lambda}}\right] \quad (41)$$

Using Equation (33) or (35) Equation (41) can be transformed:

$$(1+\alpha)e^{\frac{x^*}{1+\alpha}} = e^{x^*} \quad (42)$$

The solution of Equation (42) can be obtained easily:

$$x^* = x^*(\alpha) = \frac{1+\alpha}{\alpha}\ln(1+\alpha) \quad (43)$$

By substituting Equation (43) into Equation (40) the mean tensile strength of the fiber system modeled with ES2-bundle can be calculated:

$$FH_2^* = FH_2(z*) = \begin{cases} \frac{(1+\alpha)}{\alpha}\frac{\lambda}{2}\left(e^{-\frac{x^*(\alpha)}{1+\alpha}} - e^{-x^*(\alpha)}\right) = \frac{\lambda}{2}(1+\alpha)^{-\frac{1}{\alpha}}, & \lambda < \frac{2}{x^*(\alpha)} = \frac{2}{\ln(1+\alpha)^{\frac{1+\alpha}{\alpha}}} \\ \frac{(1+\alpha)}{\alpha}\frac{\lambda}{2}\left[e^{-\frac{2}{\lambda(1+\alpha)}} - e^{-\frac{2}{\lambda}}\right], & \lambda \ge \frac{2}{x^*(\alpha)} = \frac{2}{\ln(1+\alpha)^{\frac{1+\alpha}{\alpha}}} \end{cases} \quad (44)$$

In Figure 10 the graphic representation of Equation (44) can be seen. The pair of the continuous curves (α = 0 and α = 1) referring to the ES2-bundle forms a similar but narrower range and provides smaller strength values than that for the ES1-bundle because of the decreasing slippage resistance.

Discussion and Generalization of Results

General Formulation of the Tensile Strength vs. Fiber Length Relationship

On the basis of the results for exponential fiber length distribution a general formulation valid for both ES1 and ES2 bundles and similar to Equation (23) can be given as a summary. The general form of the normalized mean tensile force process is as follows (i = 1 for ES1, i = 2 for ES2):

$$FH_i(z) = \begin{cases} \frac{\lambda}{2}f_{i1}\left(\frac{2z}{\lambda}, \alpha\right), & 0 \le z \le 1 \\ \frac{\lambda}{2}f_{i2}\left(\frac{2z}{\lambda}, \frac{2}{\lambda}, \alpha\right), & 1 < z \le 1+\alpha \\ 0, & 1+\alpha < z \end{cases} \quad (45)$$

The mean tensile strength of the fiber system is given by:

$$FH_i* = FH_i(z*) = \begin{cases} \frac{\lambda}{2}f_{i1}\left(\frac{2z^*}{\lambda}, \alpha\right), & z^* < 1 \\ \frac{\lambda}{2}f_{i1}\left(\frac{2}{\lambda}, \alpha\right), & z^* \ge 1 \end{cases} = \begin{cases} \frac{\lambda}{2}f_{i1}(x^*(\alpha), \alpha), & \lambda < \frac{2}{x^*(\alpha)} \\ \frac{\lambda}{2}f_{i1}\left(\frac{2}{\lambda}, \alpha\right), & \lambda \ge \frac{2}{x^*(\alpha)} \end{cases} \quad (46)$$

where $\lambda = \bar{l}/l_S$ and $x^* = 2z^*/\lambda$ is the solution of the following equation depending on α only:

$$x^* = x^*(\alpha) : \frac{df_{i1}(x)}{dx} = 0 \quad (47)$$

It is obvious that the first part of the curve given by Equation (41) is a straight

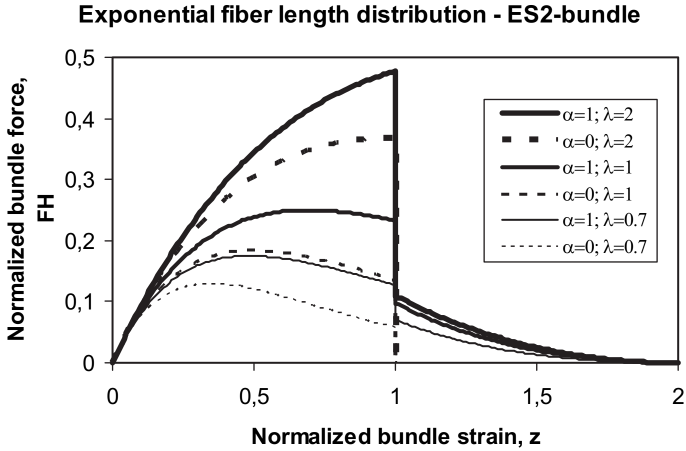

Figure 9.
Normalized tensile load versus normalized strain at different slippage factors and mean fiber length in case of ES2-bundle.

line in all cases while the second part is a curve joining the first part and tending to 1 if $\lambda \to \infty$.

Taking into account the general form of tensile strength for constant fiber length according to Equations (23) and (24) as well, Equation (46) can be generalized further on (i = 1, 2):

$$FH_i* = f_i(\lambda,\alpha) = \begin{cases} \frac{\lambda}{2} G_i(\alpha), & \lambda < \frac{2}{\mu(\alpha)} \\ H_i(\lambda,\alpha), & \lambda \geq \frac{2}{\mu(\alpha)} \end{cases} \tag{48}$$

where

$$G_i(\alpha) = \begin{cases} f_{i1}(x^*(\alpha),\alpha), & \text{exponential FLD} \\ \frac{1}{2}C(\alpha), & \text{constant FL} \end{cases} \tag{49}$$

and

$$H_i(\lambda,\alpha) = \begin{cases} \frac{\lambda}{2} f_{i1}\left(\frac{\lambda}{2},\alpha\right), & \text{exponential FLD} \\ 1 - \frac{1}{C(\alpha)\lambda}, & \text{constant FL} \end{cases} \tag{50}$$

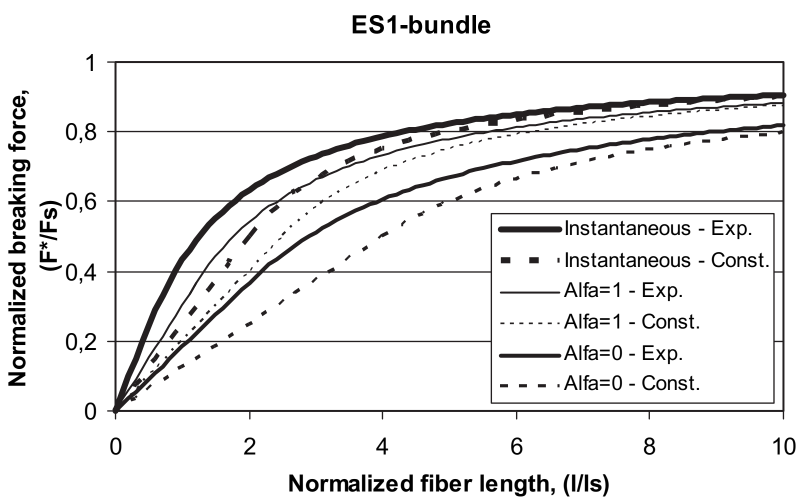

Figure 10.
Normalized breaking force modeled with ES2-bundle versus mean fiber length at different slippage factor in case of exponential FDL and constant FL.

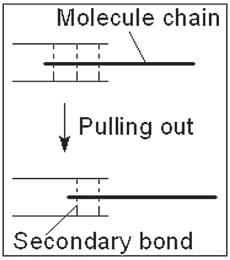

Figure 11.
Slippage of a molecule chain out of its vicinity.

finally

$$\mu(\alpha) = \begin{cases} x^*(\alpha), & \text{exponential FLD} \\ C(\alpha), & \text{constant FL} \end{cases} \quad (51)$$

Tensile Strength Estimation in Case of Finite Fiber Length Deviation

Assume that the expected value ($E(l) = \bar{l}$) and the standard deviation ($D(l) = \sigma_l$) of the fiber length are finite values and the fiber length distribution is unknown. The variance coefficient is V_l:

$$V_l = \frac{\sigma_l}{\bar{l}} \quad (52)$$

It was seen that the constant fiber length ($\bar{l} = l_o,\ V_l = 0$) and the exponential fiber length distribution ($V_l = 1$) are the extreme cases of the Erlang distributions having the same expected values as well as the tensile strength (f) of unidirectional fiber systems calculated with exponential distribution (f_{exp}) for a given damage mode (ES1 or ES2, and α) is greater than that computed with constant fiber length (f_{const}) if the normalized fiber length is finite ($0 < \lambda = \bar{l}/l_S < \infty$). The strength values are identical if $\lambda = 0$ or $\lambda \to \infty$. It is obvious that the application of convex linear combination of the two results as a kind of weighted average gives a possibility to estimate the tensile strength of fiber systems if $0 < V_l < 1$:

$$f(\lambda, \alpha) = \frac{F^*(\lambda, \alpha)}{F_S} = w(\alpha) f_{\exp}(\lambda, \alpha) + (1 - w(\alpha)) f_{const}(\lambda, \alpha) \quad (53)$$

where $0 \le w(\alpha) \le 1$ is the weighting factor.

When there is no other information then e.g. the following weighting factor can be applied:

$$0 \le w(\alpha) = V_l \le 1 \quad (54)$$

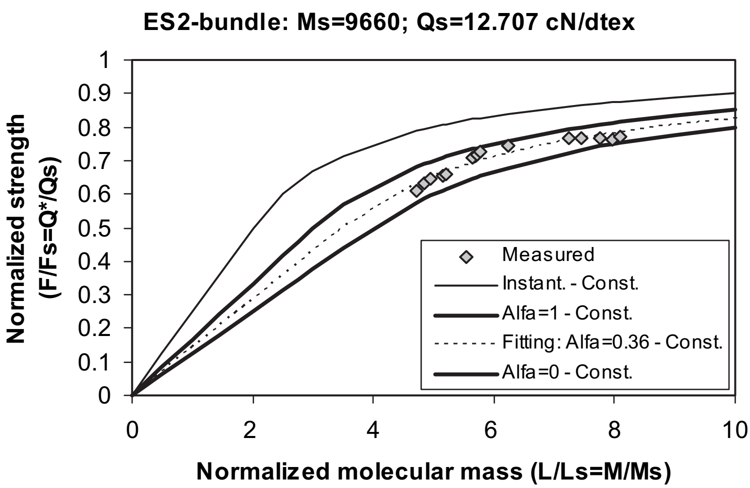

Figure 12.
Normalized strength of PP fibers versus normalized molecule mass and the fitted curve in case of constant fiber length.

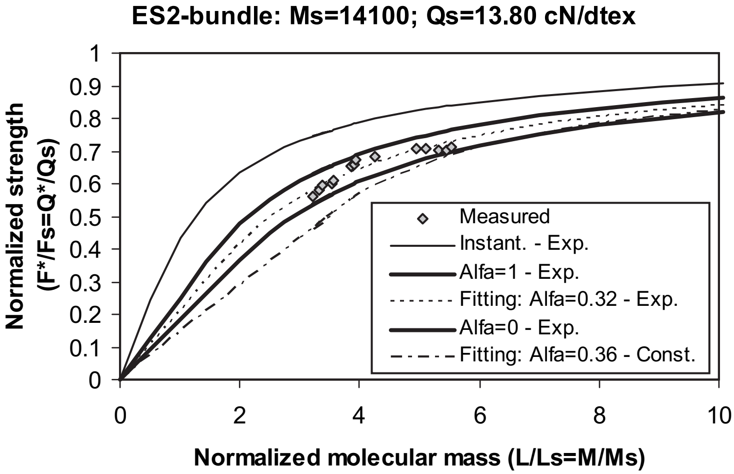

Figure 13.
Normalized strength of PP fibers versus normalized molecule mass and the fitted curve in case of exponential fiber length distribution and for comparison the fitted curve for constant fiber length.

and with that we obtain:

$$f(\lambda,\alpha) = V_l f_{\exp}(\lambda,\alpha) + (1 - V_l) f_{const}(\lambda,\alpha) \qquad (55)$$

Note that if $V_l > 1$ then the mixture of two different Erlang distributions can be used.

Application to Modeling PP Fiber Strength vs Molecular Mass

In order to demonstrate the applicability of the results let us consider oriented polymer fibers built up of molecule chains which can be considered as elementary fibers. Pulling out a molecule chain gripped by secondary bonds in an ordered part the number of bonds gradually decreases therefore the process can be modeled by the ES2-bundle (Figure 11).

In Figures 12 and 13 the tensile test data measured on isotactic polypropylene (PP) fibers by Geleji[14] at different number average molecular masses (M_n) which are essentially proportional to the mean molecule length as well as the theoretical ES2-curves calculated for constant fiber length (Figure 12) and exponential fiber length distribution (Figure 13) and fitted to these data can be seen in the range formed by the limit curves of the instantaneous damages and the ES2-bundle ($\alpha = 0$ and $\alpha = 1$).

For the sake of comparison the fitted curve in Figure 12 is depicted in Figure 13 as well.

From fitting Equation (12), the normalizing factors, that is the adhesion length or mass (M_S), the critical molecule mass ($M_{crit} = 2M_S$) and the mean specific strength of the molecule chains as well as the slippage length coefficient were determined (Table 1). The normalizing parameters are the basic properties of the model system. The approximating ES2-curves

Table 1.
Model parameters obtained from fitting the ES2-bundle model to tensile test data of PP fibers

Structural properties	Model parameters	
	Constant FL	Exponential FLD
Critical adhesion length/mass, M_S	9660	14100
Critical molecule mass, M_{crit}	19320	28200
Specific strength, Q_S [cN/dtex]	12.71	13.80
Slippage length coefficient, α	0.36	0.32

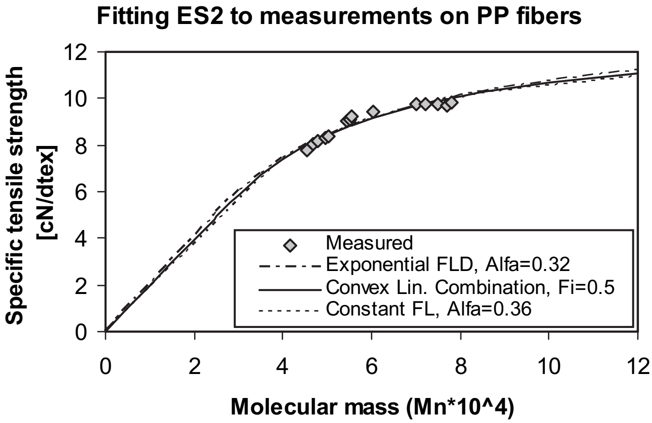

Figure 14.
Specific strength of PP fibers versus molecule mass and the fitted curves in the cases of constant fiber length and exponential fiber length distribution.

were fitted with the least square method. The goodness of fitting was found $R^2 = 0.92$ and $R^2 = 0.91$ in case of constant FL and exponential FLD, respectively.

The measured data and the normalizing factors obtained determine a model system with average properties of the PP fibers tested. The identified value of F_S corresponds to the breaking length of about 127...138 km which significantly exceeds the usual breaking length of PP fibers (32-72 km[15]), however, it is much smaller than the corresponding strength of the PP molecule chains (≈1290 km[16]). All that may be judged as real because the model system represents the fact that neither the tested PP fibers are perfectly crystalline nor the molecule chains are unidirectional.

With the aid of the parameters obtained from fittings the following relationships can be given in the original co-ordinate system for constant FL:

$$Q_c^* = \begin{cases} Q_S \frac{C(\alpha)}{4M_S} M_n, & M < \frac{2M_S}{C(\alpha)} \\ Q_S\left(1 - \frac{M_S}{C(\alpha)M_n}\right), & M \geq \frac{2M_S}{C(\alpha)} \end{cases}; \quad C(\alpha) = \frac{1+\alpha}{2+\alpha} \tag{56}$$

and exponential FLD:

$$Q_e* = \begin{cases} Q_S \dfrac{(1+\alpha)^{-1/\alpha}}{2M_S} M_n, & M < \dfrac{2M_S}{\mu(\alpha)} \\ \dfrac{Q_S}{2M_S}\dfrac{1+\alpha}{\alpha} M_n\left(e^{-\frac{2M_S}{(1+\alpha)M_n}} - e^{-\frac{2M_S}{M_n}}\right), & M \geq \frac{2M_S}{\mu(\alpha)} \end{cases} \quad \mu(\alpha) = \ln(1+\alpha)^{(1+\alpha)/\alpha} \tag{57}$$

The linear combination of Equations (56) and (58) can be formed with a given weighting factor (w):

$$Q^* = wQ_e^* + (1-w)Q_c^* \tag{58}$$

The dotted line curves according to Equation (56) and (57) depicted in Figure 14 representing the relations for fiber length distributions with variance coefficients $V_f = 0$ and $V_f = 1$ run close to each other. It means that the convex linear combination of them calculated with a weighting factor of w = 0.5 (continuous line in Figure 14) can be well used for a real case of $0 < V_f < 1$.

The maximum relative difference between the two fitted curves related to the maximum specific strength in the molecular mass range up to 120,000 is about 3.6%. Consequently this difference between the fitted curves and their convex linear combination comes to about 1.8% in this range.

Conclusion

The strength of unidirectional fibrous structures and its dependence on fiber length were modeled at different damage modes using the instantaneous fracture model and the special versions of the ES-bundle for gradual damage.

In case of exponential fiber length distribution and constant fiber breaking strain simple analytical relationships between the mean tensile strength and the fiber length were derived and compared to those for constant fiber length and written in a general form that is valid for all the damage modes discussed. The convex linear combination of the solutions for exponential fiber length distribution and constant fiber length was proposed to use for cases when the variation coefficient of the fiber length is between 0 and 1.

The practical applicability of the results was demonstrated by identifying the relationship between the tensile strength and the molecular mass of PP fibers that made it possible to estimate the critical mass of molecules and the tensile strength of the molecules without further measurements.

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Appendix

In order to find a simple approximate solution of Equation (34) let us introduce the following notation:

$$y = x\frac{\alpha}{1+\alpha} = \frac{2z}{\lambda}\frac{\alpha}{1+\alpha} \qquad \text{(A1)}$$

Using the new variable according to Equation (A1) Equation (34) can be

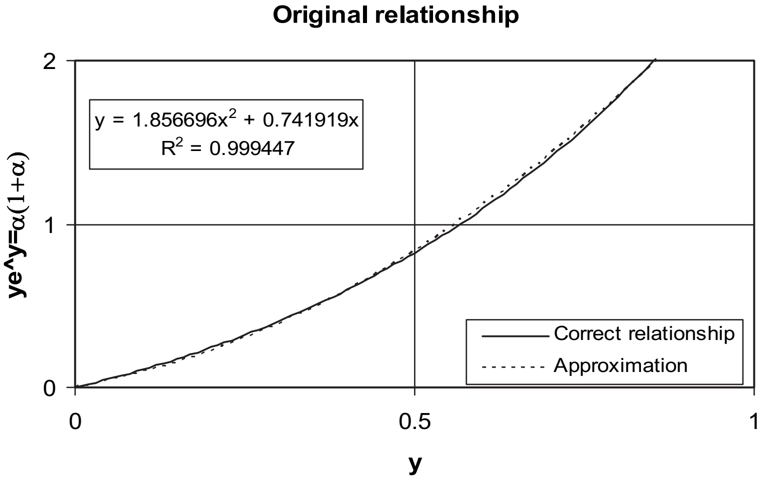

Figure A1.
Correct relationship according to Equation (A2) and its approximation.

rewritten:

$$ye^{y} = \alpha(1+\alpha) \qquad \text{(A2)}$$

Because the normal range of α is [0, 1] it is valid for the right hand side of Equation (A2) that:

$$0 \leq \alpha(1+\alpha) \leq 2 \qquad \text{(A3)}$$

The left hand side of Equation (A2) can be approximated with an easily invertible quadratic polynomial as follows:

$$\alpha(1+\alpha) = ye^{y} \approx a_o + a_1 y + a_2 y^2 \qquad \text{(A4)}$$

where $a_o = 0$ obviously. Coefficients a_1 and a_2 were determined by least squares method:

$$a_1 = 0.7419 \quad a_2 = 1.8567 \qquad \text{(A5)}$$

where the squared correlation coefficient was $R^2 = 0.9994$ (Figure A1). The maximum absolute error of this approximation came about 0.032 which means a maxi-

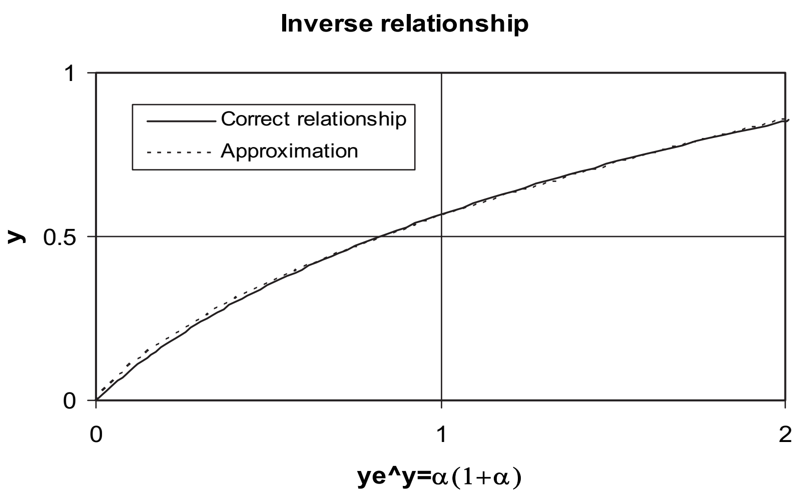

Figure A2.
Correct inverse relationship and its approximation.

mum value (2 at $\alpha = 1$) related error of 1.6%.

Taking into account Equation (A1) the approximate solution of Equation (A2) can be determined by solving the quadratic equation according to Equation (A4):

$$x^* = x^*(\alpha) = \frac{1+\alpha}{\alpha} y^* \approx \frac{1+\alpha}{\alpha} \frac{a_1}{2a_2} \left(\sqrt{1 + 4a_2 \frac{\alpha(1+\alpha)}{a_1^2}} - 1 \right) \quad \text{(A6)}$$

The inverse relationship calculated with Equation (A6) is visible in Figure A2.

The maximum absolute error for the inverse approximation was 0.016 meaning a relative error of 1.89% related to the maximum value of y (0.853).

Symbols and Abbreviations

f_b	specific adhesive resistance
l	length of fibers
l_o	constant fiber length
l_S	critical adhesion length of fibers
$l_{crit} = 2l_S$	critical length of fibers
l^+, l^-	beard lengths of fibers
l_m	active beard length of fibers
$q_l(x)$	probability density function of the fiber length
u	relative strain of the fiber bundle or fibrous structure
w	weighting factor
$x^*(\alpha)$	constant for determining the border point between the two sections of the fiber length range in case of exponential fiber length distribution
z	normalized fiber bundle strain
$C(\alpha)$	constant for determining the border point between the two sections of the fiber length range in case of constant fiber length
F	tensile force
F_b	adhesive resistance
F_S	breaking force of fibers
FH	tensile force of the fibrous structure referred to one fiber and the breaking force of a single fiber
F^*	tensile strength of the fibrous structure referred to one fiber
K	tensile stiffness of fibers
M	number average molecule mass
M_S	adhesion molecule mass corresponding to the adhesion molecule length
P	measure of probability
Q	specific tensile strength of the fibrous structure
Q_S	specific tensile strength of fibers
$Q_l(x)$	distribution function of the fiber length
S(x)	distribution function of the fiber beard length
$S_m(x)$	distribution function of the active fiber beard length
V_l	variation coefficient of fiber length
$\mathbf{1}(x)$	unit step function
α	slippage length coefficient
ε_b	relative strain of fibers corresponding to the adhesive resistance
ε_S	relative breaking strain of fibers
$\mu(\alpha)$	constant for determining the border point between the two sections of the fiber length range in the general case
λ	normalized mean fiber length
EFLD	exponential fiber length distribution
ES	ES-bundle the fibers of which break or slip out
ES1, ES2	special versions of the ES-bundle
FBL	fiber beard length
FL	fiber length
FLD	fiber length distribution
HPPE	High Performance Polyethylene
LCP	Liquid Crystal Polymer
PP	polypropylene

